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ON THE FORM AND POSITION OF THE SEA-LEVEL AS DEPENDENT ON SUPERFICIAL MASSES SYMMETRICALLY DISPOSED WITH RESPECT TO A RADIUS OF THE EARTH'S SURFACE.

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1. A considerable class of problems concerning the sea-level is that in which the attracting or disturbing mass is symmetrically disposed about a radius of the earth's surface and is situated on or near that surface. Examples of such masses, proximate at least, are portions of the earth's crust, like a continental table-land, and the ice-caps which are supposed to have been of great extent and thickness during certain epochs of the earth's history.

It is proposed in the following paper to develop the theory of the solution of this class of problems so far as is necessary to render practicable the numerical evaluation of the characteristic effects of the disturbing mass in any special case.

2. The solution of all problems of the kind outlined above depends on the principle of hydrostatics that the potential of the forces producing a free liquid surface in equilibrium has a constant value for all points of that surface. In the case of the earth, if the potential of all the attractive forces acting on a unit mass at any point of the sea-surface be denoted by P, the distance of the point from the axis of rotation by l and the velocity of rotation by ω, the form of the surface will be completely defined by the equation

$$P + \tfrac{1}{2}l^2\omega^2 = \text{a constant.} \tag{1}$$

The surface defined by this equation is now commonly called the *geoid*. It, as represented by the ocean surface, is a real surface and does not necessarily coincide with the earth's ellipsoid, which is an ideal surface of reference.

3. The exact value of P in (1) is a complicated function of the densities of the element-particles of the earth and of the co-ordinates of those particles and the attracted point. For the present purposes, however, it will suffice to consider P due to a centrobaric sphere of equal mass and volume with the earth

and concentric with the earth's centre of gravity. Since we shall only deal with relative positions of any point on the sea-surface, the potential due to centrifugal force, which is represented by the second term in (1), may be neglected.

If a and b denote the equatorial and polar semi-axes respectively of the earth's ellipsoid, and r_0 the radius of the sphere just referred to,

$$r_0 = \sqrt[3]{a^2b}. \tag{2}$$

The known values of a and b (Clarke, 1866) give $r_0 = 20902394$ English feet, or in round numbers 21000000 feet. The surface of the sphere thus defined may be regarded as the surface assumed by a thin film of sea-water covering a nucleus whose mass plus the mass of the film equals the earth's mass. We shall call this ideal surface the undisturbed surface. With respect to it the real surface of the earth lies partly without and partly within; but, so far as small relative changes in sea-level are concerned, it is practically immaterial whether we refer to the actual, closely spheroidal surface or to the simpler spherical one.

4. Let $M =$ mass of the earth,
$\rho_m =$ mean density of the earth.

Then

$$M = \tfrac{4}{3} r_0^3 \pi \rho_m; \tag{3}$$

and the equation to the undisturbed surface is

$$\frac{M}{r_0} = \tfrac{4}{3} r_0^2 \pi \rho_m = C_1, \text{ a constant.} \tag{4}$$

Suppose now a new mass m, of density ρ (positive or negative), placed in any fixed position relative to the undisturbed surface. The resulting new sea-surface will then differ from that defined by (4). To determine this difference, let V be the potential of the disturbing mass m at any point of the disturbed surface, and let v denote the elevation or depression of this point with respect to the undisturbed surface. The equation of the disturbed surface will then be

$$\frac{M}{r_0 + v} + V = C_2, \text{ a constant.} \tag{5}$$

The difference of this and (4), to terms of the first order inclusive in v, is

$$-\frac{M}{r_0^2} v + V = C_2 - C_1;$$

whence, putting

$$V_0 = C_2 - C_1,$$

we have

$$v = (V - V_0)\frac{r_0^2}{M}. \tag{6}$$

Since (if we omit the unit of attraction, which disappears by division from

our expression) $M/r_0^2 = g$, the velocity-increment at the surface of the earth due to the earth's attraction, (6) may be written

$$v = \frac{V - V_0}{g}. \tag{6'}$$

5. V_0 in the last two equations is the value of V where $v = 0$, or the value of V along the line of intersection of the disturbed and undisturbed surfaces. If we put

$$v_0 = V_0 \frac{r_0^2}{M}.$$

$$v + v_0 = V \frac{r_0^2}{M} = \frac{V}{g}. \tag{7}$$

This equation represents the elevation of the disturbed surface above a spherical surface of equal potential, whose value is

$$\frac{M}{r_0 - v} = C_2,$$

since the difference between this and (5) gives (7).

The constant V_0 may be determined from the obvious condition that the disturbed and undisturbed surfaces contain equal volumes. It should be observed also in this connection, that V_0 is never of a lower order than V, and cannot therefore be neglected in comparison with V. This remark is important since some writers* have neglected V_0 and arrived at the erroneous equation

$$v = \frac{V}{g}$$

instead of (6′) or (7).

6. It is evident that the equations derived above, (5) to (7), will hold true if the disturbing mass m be a part of the earth's mass so long as the ratio m/M may be neglected in comparison with unity. Thus, m may represent the mass of a continent, the deficiency in mass of a lake or lake-basin, etc.

7. The next step requires the determination of the potential V of the attracting mass for any point of the disturbed surface, whether without or within the circle which we have assumed to define the boundary of the mass.

In order to derive an expression for V, let the rectangular and polar co-ordinates of any point of the disturbing mass be defined by the usual relations, viz.:—

$$x = r \cos\theta \cos\lambda,$$
$$y = r \cos\theta \sin\lambda,$$
$$z = r \sin\theta;$$

*Notably Archdeacon Pratt. See his *Figure of the Earth*, 4th edition, articles 200 and 213.

in which θ and λ correspond to polar distance and longitude respectively, the position of the origin being arbitrary. With reference to the same origin, let the co-ordinates of the attracted point on the sea-surface be

$$\begin{aligned} x' &= r' \cos\theta' \cos\lambda', \\ y' &= r' \cos\theta' \sin\lambda', \\ z' &= r' \sin\theta'. \end{aligned}$$

If D denote the distance between the attracting and attracted points and

$$\cos\psi = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\lambda - \lambda'), \tag{8}$$

$$D^2 = r^2 + r'^2 - 2rr'\cos\psi = (r - r')^2 + 4rr'\sin^2\tfrac{1}{2}\psi. \tag{9}$$

The volume-element of the attracting mass is

$$dx\,dy\,dz = r^2 dr \sin\theta\, d\theta\, d\lambda.$$

Hence if ρ denote the density, supposed uniform, of the attracting mass, a general expression for the required potential is

$$V = \rho\int\int\int\frac{r^2 dr \sin\theta\, d\theta\, d\lambda}{D}. \tag{10}$$

We must now evaluate this integral. Taking the centre of the sphere of reference as origin of co-ordinates, let

$$\begin{aligned} r &= r_0 + u \\ \text{and} \qquad r' &= r_0 + v, \end{aligned} \tag{11}$$

in which u and v are small quantities relative to r_0, v being the same of course as defined by (6). Premising what will be proved hereafter, namely, that quantities of the order u/r_0, v/r_0, and $(u-v)^2/r_0^2$ may be neglected, equation (9) gives

$$D = 2r_0\sin\tfrac{1}{2}\psi \tag{12}$$

From the first of (11)

$$\begin{aligned} dr &= du, \\ \text{and} \qquad r^2 &= r_0^2, \end{aligned} \tag{13}$$

to terms of the order u/r.

Without loss of generality we may assume the line from which θ and θ' are reckoned to pass through the attracted point, and the plane from which λ and λ' are reckoned to pass through the attracted point and the centre of the attracting mass. In this case $\theta' = 0$, and $\lambda' = 0$, and (8) gives

$$\psi = \theta.$$

By means of this relation and the values in (12) and (13) equation (10) becomes

$$V = r_0 \rho \int\int\int du \cos \tfrac{1}{2}\theta \, d\theta \, d\lambda. \tag{14}$$

Assuming for the present that the attracting mass is of uniform thickness, h, the limits of u will be 0 and h. Let the limits of θ, which are obviously functions of λ, be θ_1 and θ_2. The limits of λ are evidently equal in magnitude, but of opposite signs. Hence we have

$$V = 2r_0\rho \int_0^h \int_{\theta_1}^{\theta_2} \int_0^\lambda du \cos \tfrac{1}{2}\theta \, d\theta \, d\lambda = 4r_0 h\rho \int_0^\lambda (\sin \tfrac{1}{2}\theta_2 - \sin \tfrac{1}{2}\theta_1) \, d\lambda. \tag{15}$$

To complete the evaluation of (15) it will be convenient to change variables. Consider the spherical triangles formed by the attracted and attracting points, the centre of the attracting mass, and the points in which the arc θ cuts the circle bounding the mass. Thus, in figures 1 and 2 let P be the attracting and A the attracted points, C the centre of the attracting mass and BDG the bounding circle. Then,

$$\theta = AP$$

and
$$\lambda = BAC.$$

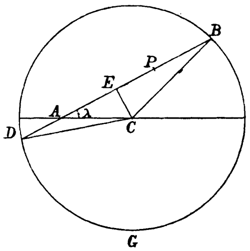

Fig. 1.

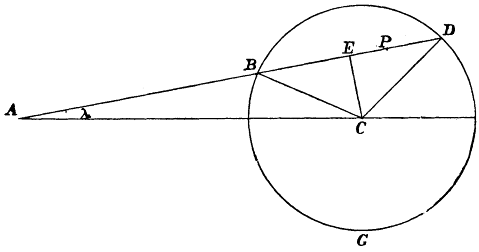

Fig. 2.

Draw CE perpendicular to AB and put

$$AC = \alpha, \qquad BC = \beta,$$
$$PE = s, \qquad BE = s_0,$$
$$CE = p, \qquad AE = q.$$

From either figure

$$\theta = q + s,$$
$$\theta_1 = q - s_0,$$
$$\theta_2 = q + s_0;$$

whence

$$\sin \tfrac{1}{2}\theta_2 - \sin \tfrac{1}{2}\theta_1 = 2 \cos \tfrac{1}{2}q \sin \tfrac{1}{2}s_0. \tag{16}$$

The right-angled triangles of either figure give

$$\left.\begin{aligned}\cos q &= \frac{\cos \alpha}{\cos p},\\ \cos s_0 &= \frac{\cos \beta}{\cos p},\\ \sin p &= \sin \alpha \sin \lambda.\end{aligned}\right\} \tag{17}$$

The first two of (17) give

$$2 \cos^2 \tfrac{1}{2} q = 1 + \frac{\cos \alpha}{\cos p},$$

$$2 \cos^2 \tfrac{1}{2} s_0 = 1 - \frac{\cos \beta}{\cos p};$$

whence $$2 \cos \tfrac{1}{2}q \sin \tfrac{1}{2}s_0 = \frac{\sqrt{[(\cos p + \cos \alpha)(\cos p - \cos \beta)]}}{\cos p}. \tag{18}$$

From the last of (17)

$$d\lambda = \frac{\cos p \, dp}{\sqrt{(\cos^2 p - \cos^2 \alpha)}}. \tag{19}$$

Now the last of equations (17) and the diagrams show that the limits of p corresponding to the limits of λ are 0 and α, or 0 and β, according as the attracted point is within or without the circle bounding the attracting mass. Hence, if we denote the potentials in the two cases by V_1 and V_2 respectively, the equivalents in (15), (16), (18), and (19) give

$$V_1 = 4r_0 h\rho \int_0^{\alpha} \sqrt{\left(\frac{\cos p - \cos \beta}{\cos p - \cos \alpha}\right)} \,.\, dp, \tag{20}$$

$$\alpha \leqq \beta$$

$$V_2 = 4r_0 h\rho \int_0^{\beta} \sqrt{\left(\frac{\cos p - \cos \beta}{\cos p - \cos \alpha}\right)} \,.\, dp. \tag{21}$$

$$\alpha \geqq \beta$$

8. The integrals in these equations are in general elliptics of the third species. They may be evaluated by the usual processes applicable to elliptics, by mechanical quadrature, or by series. The integral in (20) presents some apparent difficulty, since the element-function is infinite at the upper limit except in the case $\alpha = \beta$. Again in case $\alpha = 0$, the integral assumes this anomalous form

$$\int_0^0 \sqrt{\left(\frac{1 - \cos \beta}{1 - 1}\right)} \,.\, dp,$$

the value of which is $\pi \sin \frac{1}{2}\beta$, as may be easily verified by means of (15), (16), and (18). These peculiar features may be removed by the following transformation, which secures the same constant limits for both (20) and (21).

For brevity put

$$I_1 = \int_0^{\alpha} \sqrt{\left(\frac{\cos p - \cos \beta}{\cos p - \cos \alpha}\right)} \cdot dp, \tag{22}$$

$$I_2 = \int_0^{\beta} \sqrt{\left(\frac{\cos p - \cos \beta}{\cos p - \cos \alpha}\right)} \cdot dp. \tag{23}$$

Then, observing that $\dfrac{\cos p - \cos \beta}{\cos p - \cos \alpha} = \dfrac{\sin^2 \frac{1}{2}\beta - \sin^2 \frac{1}{2}p}{\sin^2 \frac{1}{2}\alpha - \sin^2 \frac{1}{2}p}$,

put in I_1 $\qquad \sin \frac{1}{2}p = \sin \frac{1}{2}\alpha \sin \gamma_1,$

and in I_2 $\qquad \sin \frac{1}{2}p = \sin \frac{1}{2}\beta \sin \gamma_2.$

These give

$$dp = \frac{2 \sin \frac{1}{2}\alpha \cos \gamma_1 \, d\gamma_1}{\sqrt{(1 - \sin^2 \frac{1}{2}\alpha \sin^2 \gamma_1)}},$$

$$dp = \frac{2 \sin \frac{1}{2}\beta \cos \gamma_2 \, d\gamma_2}{\sqrt{(1 - \sin^2 \frac{1}{2}\beta \sin^2 \gamma_2)}},$$

and the limits for both γ_1 and γ_2 are 0 and $\frac{1}{2}\pi$. Therefore

$$I_1 = \int_0^{\frac{1}{2}\pi} \frac{2 \sin \frac{1}{2}\beta \sqrt{\left(1 - \dfrac{\sin^2 \frac{1}{2}\alpha}{\sin^2 \frac{1}{2}\beta} \sin^2 \gamma_1\right)} \, d\gamma_1}{\sqrt{(1 - \sin^2 \frac{1}{2}\alpha \sin^2 \gamma_1)}}, \tag{24}$$

$$\alpha \leqq \beta$$

$$I_2 = \int_0^{\frac{1}{2}\pi} \frac{2 \sin^2 \frac{1}{2}\beta \cos^2 \gamma_2 \, d\gamma_2}{\sin \frac{1}{2}\alpha \sqrt{\left(1 - \dfrac{\sin^2 \frac{1}{2}\beta}{\sin^2 \frac{1}{2}\alpha} \sin^2 \gamma_2\right)} \cdot \sqrt{(1 - \sin^2 \frac{1}{2}\beta \sin^2 \gamma_2)}}. \tag{25}$$

$$\alpha \geqq \beta$$

[TO BE CONTINUED.]